J. Fluid Mech. (1977), *vol.* 82, *part* 3, *pp.* 401–413
Printed in Great Britain

Formation of the hexagonal pattern on the surface of a ferromagnetic fluid in an applied magnetic field

By A. GAILĪTIS
Institute of Physics of the Latvian Academy of Sciences, Riga, USSR

(Received 19 March 1970)

When a ferromagnetic fluid with a horizontal free surface is subjected to a uniform vertical applied magnetic field B_0, it is known (Cowley & Rosensweig 1967) that the surface may be unstable when the field strength exceeds a certain critical value B_c. In this paper we consider, by means of an energy minimization principle, the possible forms that the surface may then take. Under the assumption that $|\mu - 1| \ll 1$ (where μ is the magnetic permeability of the fluid), it is shown that when B_0 is near to B_c there are three equilibrium configurations for the surface: (i) flat surface, (ii) stationary hexagonal pattern, (iii) stationary square pattern. Configuration (i) is stable for $B_0 < B_c$, (ii) is stable for $B_0 > B_c$ and $B_0 - B_c$ sufficiently small, and (iii) is stable for some higher values of B_0. In each configuration the fluid is static, and the surface is in equilibrium under the joint action of gravity, surface tension, and magnetic forces. The amplitude of the surface perturbation in cases (ii) and (iii) is calculated, and hysteresis effects associated with increase and decrease of B_0 are discussed.

1. Introduction

In Cowley & Rosensweig's (1967) experiments a stationary wave pattern was observed on the surface of a ferromagnetic fluid subjected to a vertical magnetic field. The crests of the pattern formed a hexagonal array (in one test such an array transformed into a square array). A similar picture is well known in the context of thermal convection between two horizontal planes, where the velocity field divides into hexagonal cells (Bénard 1901). The explanations of the two phenomena are similar. While the magnetic field (or temperature gradient) remains small, a horizontal surface (or immobile fluid) represents the stable equilibrium state of the system. When the field (or gradient) grows and exceeds the critical magnitude at which this equilibrium becomes unstable, the surface takes a more stable form (or convection starts). For the theoretical calculation of the critical field (Frenkel 1935; Melcher 1963; Cowley & Rosensweig 1967) or temperature gradient (Rayleigh 1916; Pellew & Southwell 1940) the deviations from equilibrium may be assumed small and the equations may be linearized. Linearized equations are successful also for the calculation of the critical wavenumber but they are inadequate for the full description of either phenomenon. The symmetry of the developed wave array (hexagonal, square or, possibly, of some other form) and the wave amplitudes may be determined only from nonlinear equations. For the calculations of wave amplitudes, nonlinear equations were used by Zaitsev & Shliomis (1969) but for one-dimensional waves only. In a previous paper (Gailītis 1969) we showed that at the critical field the main mode is the hexagonal array, but there was no attempt to determine the amplitude.

This article contains a more complete treatment of the problem. The method adopted is similar to that developed for the thermal convection problem by Palm (1960) and Segel & Stuart (1962). However, for the problem under consideration we use an energy variational principle rather than proceeding from the equations of motion.

The following text contains four sections. In §2, the general form for the lowest terms in the potential energy expansion is established. In §3, the coefficients appearing in this expansion are calculated. In §4, the potential energy is minimized and the corresponding wave amplitudes are calculated. Depending on the field level, the energy minimum is given by one of three possible surface forms: unperturbed flat surface, square wave array and hexagonal wave array. The last type is described by the same solution as the hexagonal convection cell in the paper of Segel & Stuart (1962). We shall use a more general form for the surface equation than that used for the cell in the cited paper. Therefore another solution from this paper corresponds to an unstable equilibrium configuration of the surface.

Finally §5 contains the conclusions and a discussion about hysteresis phenomena in transitions from one surface configuration to another.

We may use the lowest terms in a power series instead of the exact expression if two conditions are satisfied: (i) the external field B_0 must be close to the critical value B_c and (ii) the permeability μ must be close to unity. Therefore the problem contains two small parameters, $(\mu-1)^2$ and $\epsilon = B_0^2/B_c^2 - 1$, and the phenomenon depends on the ratio $\epsilon/(\mu-1)^2$, which may be of any magnitude.

All that is said about ferromagnetic fluid in a magnetic field applies also to the problem of a dielectric fluid in an electric field with one additional condition: the electrical resistance of the fluid must be high enough to prevent accumulation of free charges on the surface in the experimental time (observations in the opposite situation were reported by Taylor & McEwan 1965).

2. Formulation

Consider an infinite horizontal interface between an incompressible ferromagnetic liquid (μ = constant > 1) and a vacuum ($\mu = 1$) in a vertical external magnetic field B_0 and gravitational field g. When $B_0 = 0$, the only stable equilibrium interface is a flat horizontal surface, which may be taken to be the x, y plane. The z axis is directed upwards (from liquid to vacuum). For the time being we suppose that the surface has an arbitrary form $z = \zeta(x, y)$, and we calculate the potential energy of the whole system (per unit area of the unperturbed surface):

$$\mathscr{U}(\zeta) = \tfrac{1}{2}\rho g \overline{\zeta^2(x,y)} + \alpha \overline{[1 + (\operatorname{grad} \zeta(x,y))^2]^{\frac{1}{2}}} + \overline{\frac{1}{2\mu_0}\int_{-\infty}^{\infty} \frac{B^2(x,y,z)}{\mu(x,y,z)}\,dz}. \qquad (1)$$

The overbar denotes an average over the whole x, y plane:

$$\overline{F(x,y)} = \lim_{S\to\infty} S^{-1} \iint_S F(x,y)\,dx\,dy.$$

The first term in (1) is hydrostatic energy (ρ is the density of the fluid), the second is surface energy (α is the surface tension) and the last term is the magnetic energy (if $z < \zeta(x,y)$ then $\mu(x,y,z) = \mu$, otherwise $\mu(x,y,z) = 1$). All the expressions may be simplified if the energy $\mathscr{U}$ is measured in units of α and all linear dimensions in units of $(\alpha/\rho g)^{\frac{1}{2}}$. On this dimensionless scale the critical wavenumber is 1, and equation (1) may be rewritten as

$$\mathscr{U}(\zeta) = \tfrac{1}{2}\overline{\zeta^2(x,y)} + \overline{[1+(\operatorname{grad}\zeta(x,y))^2]^{\frac{1}{2}}} + \frac{(\alpha\rho g)^{-\frac{1}{2}}}{2\mu_0}\overline{\int_{-\infty}^{\infty}\frac{B^2(x,y,z)}{\mu(x,y,z)}dz}. \tag{2}$$

The surface $\zeta(x,y)$ is now represented as a superposition of M different ($\boldsymbol{\kappa}_i \neq \pm\boldsymbol{\kappa}_j$ if $i \neq j$) one-dimensional waves. This superposition is compiled from N main waves with critical (unit) wavenumbers ($|\boldsymbol{\kappa}_i| = 1, i \leqslant N$) and from $M-N$ harmonics of these main waves (additional waves):

$$\begin{aligned}\zeta(x,y) &= \sum_{i=1}^{M} a_{\kappa_i}\cos(\boldsymbol{\kappa}_i\,.\,\mathbf{r}+\delta_i)\\ &= \sum_{i=1}^{N} a_{\kappa_i}\cos(\boldsymbol{\kappa}_i\,.\,\mathbf{r}+\delta_i) + \sum_{i=1}^{N} a_{2\kappa_i}\cos(2\boldsymbol{\kappa}_i\,.\,\mathbf{r}+2\delta_i)\\ &\quad+\sum_{\pm}\sum_{\substack{i<j\leqslant N\\ |\kappa_i\pm\kappa_j|\neq 1}} a_{\kappa_i\pm\kappa_j}\cos((\boldsymbol{\kappa}_i\pm\boldsymbol{\kappa}_j)\,.\,\mathbf{r}+\delta_i+\delta_j).\end{aligned} \tag{3}$$

The two forms for $\zeta(x,y)$ show the choice of wave vectors and phases for the additional waves. The directions of the main wave vectors for the time being are arbitrary with one exception: if there is any pair (i,j) of main waves with vector sum or difference equal to the unit vector ($|\boldsymbol{\kappa}_i \pm \boldsymbol{\kappa}_j| = 1$) then there must also be included among the main waves the wave with vector $\boldsymbol{\kappa}_i \pm \boldsymbol{\kappa}_j$ (or with the opposite vector $-\boldsymbol{\kappa}_i \mp \boldsymbol{\kappa}_j$). Such a wave must not be included among the additional waves. The amplitudes of all the waves, the directions of the main vectors, the phases of the main waves, and the number N of terms in the sum (3) are for the time being arbitrary. In §4 they will be varied to provide the minimum of $\mathscr{U}(\zeta)$.

We have no explicit formula expressing $\mathscr{U}(\zeta)$ in terms of these quantities. Therefore we shall obtain an expansion of $\mathscr{U}(\zeta)$ as a series in powers of the wave amplitudes up to the square of the additional wave amplitudes and the fourth power of the main wave amplitudes inclusive. This expansion contains three functions, $E(B_0, |\boldsymbol{\kappa}|)$, $K(\theta)$ and $Q(\theta)$, which will be determined in §3. For the moment, we simply state that the following form may be obtained by symmetry considerations alone (for details, see appendix):

$$\begin{aligned}\mathscr{U}(\zeta)-\mathscr{U}(0) = &-\tfrac{1}{2}E(B_0,1)\sum_{i=1}^{N} a_{\kappa_i}^2\\ &-Q(120^\circ)\sum_{\substack{i<j<l\leqslant N\\ \kappa_i\pm\kappa_j\pm\kappa_l=0}}\cos(\delta_i\pm\delta_j\pm\delta_l)\,a_{\kappa_i}a_{\kappa_j}a_{\kappa_l}+\tfrac{1}{4}K(0)\sum_{i=1}^{N}a_{\kappa_i}^4\\ &+\sum_{i<j\leqslant N}K(\theta_{ij})\,a_{\kappa_i}^2a_{\kappa_j}^2-\frac{1}{2}\sum_{i=1}^{N}[Q(0)\,a_{\kappa_i}^2a_{2\kappa_i}+E(B_0,2)\,a_{2\kappa_i}^2]\\ &-\sum_{\pm}\sum_{\substack{i<j\leqslant N\\ |\kappa_i\pm\kappa_j|\neq 1}}[Q(90^\circ\pm\theta_{ij}\mp 90^\circ)\,a_{\kappa_i}a_{\kappa_j}a_{\kappa_i\pm\kappa_j}+\tfrac{1}{2}E(B_0,|\boldsymbol{\kappa}_i\pm\boldsymbol{\kappa}_j|)\,a_{\kappa_i\pm\kappa_j}^2]\\ &+O(a_{\kappa_i}^5).\end{aligned} \tag{4}$$

All results of the linear theory are contained in the function $E(B_0, |\boldsymbol{\kappa}|)$. In a subcritical field ($B_0 < B_c$), $E(B_0, |\boldsymbol{\kappa}|)$ is negative for all $\boldsymbol{\kappa}$, and therefore the flat surface $\zeta = 0$ is stable. In the critical field for the critical wavenumber (1 in our units),

$E(B_0, 1) = 0$ (for any other $|\boldsymbol{\kappa}| \neq 1$ it remains negative) and the flat surface is neutrally stable. In stronger fields, $E(B_0, 1)$ is positive and the flat surface is unstable.

We shall restrict attention to a range of fields close to the critical field; hence in the other functions K and Q and also in $E(B_0, |\boldsymbol{\kappa}|)$ for $|\boldsymbol{\kappa}| \neq 1$, the difference between B_0 and B_c may be neglected and B_0 may be replaced by B_c.

The amplitudes of the additional waves appear in (4) only in the form

$$A_l a_{\kappa_l} - \tfrac{1}{2} E(B_c, |\boldsymbol{\kappa}_l|)\, a^2_{\kappa_l},$$

where $|\boldsymbol{\kappa}_l| \neq 1$ and A_l depends on the amplitudes of the main waves. This form permits immediate minimization with respect to the amplitudes of the additional waves at fixed values of the main amplitudes. Denoting this partly minimized difference $\mathscr{U}(\zeta) - \mathscr{U}(0)$ by $\delta\mathscr{U}_N$ we get

$$\delta\mathscr{U}_N = -\tfrac{1}{2}\epsilon \sum_{i=1}^{N} a^2_{\kappa_i} - \gamma \sum_{\substack{\kappa_i \pm \kappa_j \pm \kappa_l = 0 \\ i<j<l\leqslant N}} \cos(\delta_i \pm \delta_j \pm \delta_l)\, a_{\kappa_i} a_{\kappa_j} a_{\kappa_l}$$
$$+ \tfrac{1}{4}\beta_0 \sum_{i=1}^{N} a^4_{\kappa_i} + \sum_{i<j\leqslant N} \beta(\theta_{ij})\, a^2_{\kappa_i} a^2_{\kappa_j}, \tag{5}$$

where

$$\left.\begin{aligned} \epsilon &= E(B_0, 1), \quad \gamma = Q(120^\circ), \\ \beta_0 &= K(0) + \tfrac{1}{2} Q^2(0)/E(B_0, 2), \\ \beta(\theta_{ij}) &= K(\theta_{ij}) + \tfrac{1}{2}[Q^2(\theta_{ij})/E(B_c, |\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j|) \\ &\quad + Q^2(180^\circ - \theta_{ij})/E(B_c, |\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j|)] \quad \text{if} \quad \theta_{ij} \neq 60^\circ, 120^\circ. \end{aligned}\right\} \tag{6}$$

If $\theta_{ij} = 60^\circ$, the term with zero denominator $E(B_c, |\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j|)$ must be omitted in the last expression; if $\theta_{ij} = 120^\circ$, the term with denominator $E(B_c, |\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j|)$ must be omitted.

3. The perturbed magnetic field

In the expansion (4), the coefficients E, K and Q are independent of the phases. Therefore in calculating them, the phases may be set equal to zero ($\delta_i = 0$). The first two terms in the energy (2) may easily be obtained in the form (4): for the first term only the average value $\overline{\zeta^2}$ need be calculated; for the second term, the square root must first be expanded in series.

Transformation of the last term is more complicated, because the magnetic field $\mathbf{B}(x, y, z)$ must be calculated. We may look for the field in the form

$$\mathbf{B}(x, y, z) = B_0 \operatorname{grad} \phi(x, y, z),$$

and we must satisfy the following conditions:

(i) $\mathbf{B}(x, y, z) \to \mathbf{B}_0$ as $z \to \pm\infty$;

(ii) $\operatorname{div} \mathbf{B} = 0$;

(iii) the dependence of ϕ on x and y should be similar to that of $\zeta(x, y)$ for the boundary conditions to be easily satisfied. For ϕ there are two different expressions, one (ϕ^-) above the surface $\zeta(x, y)$ and another (ϕ^+) below it:

$$\phi^{\pm} = \text{const}^{\pm} + z \pm \sum_{i=1}^{M} b^{\pm}_{\kappa_i} \exp(\pm \kappa_i z) \cos \boldsymbol{\kappa}_i \cdot \mathbf{r}. \tag{7}$$

The coefficients $b^{\pm}_{\kappa_i}$ are determined by the surface shape through the boundary conditions

$$\phi^+ = \mu\phi^-, \quad \frac{\partial\phi^+}{\partial z} - \frac{\partial\phi^-}{\partial z} = (\operatorname{grad}\phi^+ - \operatorname{grad}\phi^-) . \operatorname{grad}\zeta.$$

The first condition follows from the continuity of the tangential component of $\mu^{-1}B$, and the second from the continuity of the normal component of B. These boundary conditions must be applied on the surface $z = \zeta(x, y)$. Therefore after substitution of (7), the exponents must be expanded in a series. After solving the equations we have, for $i, j, l \leqslant N$,

$$b^{\pm}_{\kappa_i} = \frac{\mu-1}{\mu+1}\left[a_{\kappa_i} + \frac{\mu-1+\mu\mp 1}{2(\mu+1)} \sum_{\substack{j<l \\ \kappa_i\pm\kappa_j\pm\kappa_l=0}} a_{\kappa_j}a_{\kappa_l} + \mathscr{D}^{\pm}_i\right], \tag{8a}$$

where

$$\mathscr{D}^+_i - \mathscr{D}^-_i = -a_{\kappa_i}a_{2\kappa_i} + \sum_{\pm} \sum_{\substack{i\neq j \\ |\kappa_i\pm\kappa_j| \pm 1}} a_{\kappa_j}a_{\kappa_i\pm\kappa_j} - 2\frac{\mu-1}{\mu+1}a_{\kappa_i} \sum_{i\neq j} a^2_{\kappa_j}\left(1 - \cos^3\frac{\theta_{ij}}{2} - \sin^3\frac{\theta_{ij}}{2}\right), \tag{8b}$$

$$b^{\pm}_{2\kappa_i} = (\mu-1)(\mu+1)^{-1}[a_{2\kappa_i} \mp \tfrac{1}{2}a^2_{\kappa_i}], \tag{8c}$$

$$b^{\pm}_{\kappa_i+\kappa_j} = \frac{\mu-1}{\mu+1}\left[a_{\kappa_i+\kappa_j} + \frac{a_{\kappa_i}a_{\kappa_j}}{\mu+1}\left(\mu - 1 + (1-\mu\mp\mu\mp 1)\cos\frac{\theta_{ij}}{2}\right)\right], \tag{8d}$$

$$b^{\pm}_{\kappa_i-\kappa_j} = \frac{\mu-1}{\mu+1}\left[a_{\kappa_i-\kappa_j} + \frac{a_{\kappa_i}a_{\kappa_j}}{\mu+1}\left(\mu - 1 + (1-\mu\mp\mu\mp 1)\sin\frac{\theta_{ij}}{2}\right)\right]. \tag{8e}$$

These expressions are the leading terms of power-series expansions. The number of terms written here is sufficient for the calculation of the functions E, Q and K.

The integral in the last term of the energy (2) must be divided into two parts:

$$\int_{-\infty}^{\infty} \frac{B^2(x,y,z)}{\mu(x,y,z)}dz = B_0^2\int_{\zeta(x,y)}^{\infty} (\operatorname{grad}\phi^-)^2 dz + B_0^2\mu^{-1}\int_{-\infty}^{\zeta(x,y)} (\operatorname{grad}\phi^+)^2 dz.$$

After substituting (7) into these integrals it is easy to integrate over z. At the limits $\pm\infty$ the exponents vanish; at $z = \zeta(x, y)$, they must be expanded in series. After averaging over x and y, we get the expansion (4) with the following coefficients:

$$E(B_0, \kappa) = \epsilon - \tfrac{1}{2}(1-\kappa)^2, \quad \epsilon = B_0^2/B_c^2 - 1, \tag{9a, b}$$

$$B_c^2 = 2\mu_0(\alpha\rho g)^{\frac{1}{2}}(\mu+1)(\mu-1)^{-2}, \tag{9c}$$

$$K(\theta) = \sin^3\tfrac{1}{2}\theta + \cos^3\tfrac{1}{2}\theta - \tfrac{9}{16} - \tfrac{1}{8}\cos^2\theta + (\mu-1)^2(\mu+1)^{-2}(2 - \sin\tfrac{1}{2}\theta - \cos\tfrac{1}{2}\theta - \sin^3\tfrac{1}{2}\theta - \cos^3\tfrac{1}{2}\theta), \tag{9d}$$

$$Q(\theta) = (\mu-1)(\mu+1)^{-1}(2\cos\tfrac{1}{2}\theta - \cos^2\tfrac{1}{2}\theta). \tag{9e}$$

From (6) and (9) it follows that

$$\beta_0 = \beta(0), \quad \gamma = \tfrac{3}{4}(\mu-1)(\mu+1)^{-1}, \tag{10a, b}$$

$$\beta(\theta) = \sin^3\tfrac{1}{2}\theta + \cos^3\tfrac{1}{2}\theta - \tfrac{9}{16} - \tfrac{1}{8}\cos^2\theta + (\mu-1)^2(\mu+1)^{-2} \times\{1 - \sin\tfrac{1}{2}\theta - \cos\tfrac{1}{2}\theta - \sin^3\tfrac{1}{2}\theta - \cos^3\tfrac{1}{2}\theta - \tfrac{3}{4}\sin^2\theta[(1-2\cos\tfrac{1}{2}\theta)^{-2} + (1-2\sin\tfrac{1}{2}\theta)^{-2}]\} \quad \text{if} \quad \theta \neq 60^\circ, 120^\circ, \tag{10c}$$

$$\beta(60^\circ) = \beta(120^\circ) = -\tfrac{15}{32} + \tfrac{3}{8}\sqrt{3} + \left(\frac{\mu-1}{\mu+1}\right)^2(\tfrac{1}{16} - \tfrac{37}{32}\sqrt{3}). \tag{10d}$$

4. Various forms for the equilibrium surface

Any extremum of the expression (5) corresponds to some equilibrium form of the surface. Maxima and saddle points correspond to unstable equilibrium, while minima correspond to stable equilibrium. To find these stable equilibria, we consider the equilibrium surfaces formed by one, two and three wave modes.

4.1. *Case* $N = 1$

The energy associated with one wave a_{κ_1} is

$$\delta\mathscr{U}_1 = \tfrac{1}{2}\epsilon a_{\kappa_1}^2 + \tfrac{1}{4}\beta(0)\, a_{\kappa_1}^4.$$

It attains its minimum $\delta\mathscr{U}_{1\min}$ at $a_{\kappa_1} = a_{1\min}$, where

$$\left.\begin{array}{llll} \text{if} & \epsilon < 0, \ \beta(0) > 0 & \text{then} & a_{1\min} = 0, \quad \delta\mathscr{U}_{1\min} = 0, \\ \text{if} & \epsilon > 0, \ \beta(0) > 0 & \text{then} & a_{1\min} = (\epsilon/\beta(0))^{\frac{1}{2}}, \quad \delta\mathscr{U}_{1\min} = -\epsilon^2/(4\beta(0)). \end{array}\right\} \tag{11}$$

If $\beta < 0$, then there is no minimum.

The solution (11) and the condition $\mu < 3{\cdot}535\ldots$, which is equivalent to $\beta(0) > 0$, were found in another way by Zaitsev & Shliomis (1969). In fact the solution (11) represents an unstable surface, because within the wider class of two-wave disturbances it corresponds to a saddle point of energy (see below).

4.2. *Case* $N = 2$

In the two-wave case it is convenient to introduce two additional variables a and ψ defined by $a_{\kappa_1} = a\sin\psi$, $a_{\kappa_2} = a\cos\psi$. Then

$$\delta\mathscr{U}_2 = -\tfrac{1}{2}\epsilon a^2 + \tfrac{1}{4}[\beta(0) - \sin^2 2\psi(\tfrac{1}{2}\beta(0) - \beta(\theta_{12}))]\, a^4. \tag{12}$$

It is easy to see that the minima of the energy $\delta\mathscr{U}_2$ correspond to those angles θ_{12} for which the function $\beta(\theta_{12})$ is minimal. This function, as given by (10), is shown in figure 1 for various values of μ. There are three possible minimum points: $\theta_{12} = 60°$, $90°$, $120°$. In fact $60°$ and $120°$ are not really minimum points but points of discontinuity. There, the function $\beta(\theta_{12})$ has a finite value [see (10)] but in the limits $\theta_{12} \to 60°, 120°$, it tends to minus infinity. The origin of this behaviour is in the formal separation of all waves into main and additional modes treated differently. At $\theta_{12} = 60°$, $a_{\kappa_1-\kappa_2}$ is the amplitude of a main wave and in the energy expansion (5) is contained in the terms of all orders. At the same time, for a slightly different angle, $a_{\kappa_1-\kappa_2}$ being the amplitude of an additional wave is excluded from (5), and the corresponding energy is included in the fourth-order term with a large coefficient. Therefore in the neighbourhood of $60°$ and $120°$ the expansion (5) is inconsistent but for the precise values $60°$ and $120°$ it is valid. The only complication at these angles is that three main waves must be treated together. This is done in the next subsection (§4.3).

At $\theta_{12} = 90°$ the function $\beta(\theta_{12})$ has a minimum only if $\mu < 1{\cdot}0902\ldots$.† $\beta(90°)$ is

† This condition may not be formally treated as the existence condition for solution (14). For stability of the solution [(see 18)] the wave amplitudes must be finite (except in the limit $|\mu - 1| \ll 1$) and any such condition is influenced by higher-order terms omitted in (4).

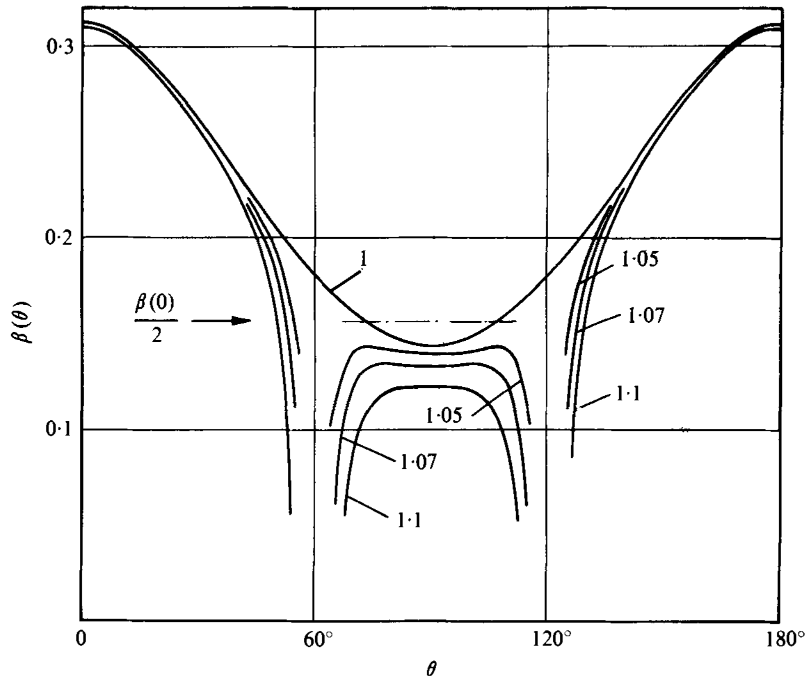

FIGURE 1. Angular dependence of the function $\beta(\theta)$ [equation (10)] for μ = 1, 1·05, 1·07 and 1·1.

then less than $\frac{1}{2}\beta(0)$ (the value $\frac{1}{2}\beta(0)$ for $\mu = 1$ is shown in figure 1 by the broken line) and so, for $\epsilon > 0$, the energy has the minimum value

$$\delta\mathscr{U}_{2\min} = -\epsilon^2(2\beta(0)+4\beta(90°))^{-1} = -0{\cdot}83\epsilon^2 \tag{13}$$†

for $\theta_{12} = 90°$ and $\sin^2 2\psi = 1$. This minimum corresponds to the occurrence of two perpendicular waves with equal amplitudes

$$|a_{\kappa_1}| = |a_{\kappa_2}| = a/\surd 2 = (\beta(0)+2\beta(90°))^{-\frac{1}{2}}\epsilon^{\frac{1}{2}} = 1{\cdot}289\epsilon^{\frac{1}{2}}. \tag{14}$$

The crests of such a superposition of two waves form a square array.

The solution (11) corresponds to $a = (\epsilon/\beta(0))^{\frac{1}{2}}$, $\theta_{12} = 90°$ and $\sin^2 2\psi = 0$. There the energy $\delta\mathscr{U}_2$ has a saddle point: as a function of a and θ_{12} it has a minimum but as a function of ψ it has a maximum. This means that the solution (11) is unstable as stated above.

4.3. *Case* $N = 3$

It was noted above that, for the system of three main waves with amplitudes a_1, a_2 and a_3 and wave vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ and $\boldsymbol{\kappa}_3$ such that $\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3 = 0$, a special treatment is needed. In this case the energy includes a third-order term involving the product of all three amplitudes:

$$\delta\mathscr{U}_3 = -\tfrac{1}{2}\epsilon(a_1^2+a_2^2+a_3^2)-\gamma\, a_1a_2a_3\cos\delta$$
$$+\tfrac{1}{4}\beta(0)\,(a_1^4+a_2^4+a_3^4)+\beta(120°)\,(a_1^2a_2^2+a_2^2a_3^2+a_3^2a_1^2),$$

† In this and subsequent formulae, numerical values are given for the limit $\mu \to 1$.

where $\delta = \delta_1 + \delta_2 + \delta_3$. From (10) it follows that $\gamma > 0$. Differentiation with respect to δ gives one of the equilibrium conditions: $\delta = 0$. The others are given by the three equations

$$\left.\begin{aligned} \epsilon a_1 + \gamma a_2 a_3 - \beta(0)\, a_1^3 - 2\beta(120^\circ)\, a_1(a_2^2 + a_3^2) &= 0, \\ \epsilon a_2 + \gamma a_3 a_1 - \beta(0)\, a_2^3 - 2\beta(120^\circ)\, a_2(a_3^2 + a_1^2) &= 0, \\ \epsilon a_3 + \gamma a_1 a_2 - \beta(0)\, a_3^3 - 2\beta(120^\circ)\, a_3(a_1^2 + a_2^2) &= 0. \end{aligned}\right\} \tag{15}$$

Defining T and W by

$$T = \beta(0) + 4\beta(120^\circ) = 1{\cdot}0356,$$
$$W = (\epsilon - \gamma^2\beta(0)(\beta(0) - 2\beta(120^\circ))^{-2})\,(\beta(0) + 2\beta(120^\circ))^{-1} = 1{\cdot}48\epsilon - 192{\cdot}8\gamma^2,$$

the system (15) gives four types of solution.

(I) Undisturbed surface, $a_1 = a_2 = a_3 = \delta\mathscr{U}_3 = 0$, which represents the energy minimum if $\epsilon < 0$.

(II) The solution (11),

$$a_1 = a_{1\min}, \quad a_2 = a_3 = 0, \quad \delta\mathscr{U}_3 = \delta\mathscr{U}_{1\min},$$

which does not represent a minimum of the full energy.

(III) Rectangular waves,

$$a_1 = \gamma/(2\beta(120^\circ) - \beta(0)), \quad a_2 = a_3 = W^{\frac{1}{2}},$$
$$\delta\mathscr{U}_3 = (4\beta(0))^{-1}[-\epsilon + (4\beta^2(120^\circ) - \beta^2(0))\, W^2],$$

which do not represent a minimum of $\delta\mathscr{U}_3$.

($\mathrm{IV}^{\pm}$) Arrays of hexagonal waves:

$$\left.\begin{aligned} a_1 = a_2 = a_3 &= (\gamma \pm (\gamma^2 + 4\epsilon T)^{\frac{1}{2}})\,(2T)^{-1}, \\ \delta\mathscr{U}_3 &= -\tfrac{3}{2}\epsilon a_1^2 - \gamma a_1^3 + \tfrac{3}{4}T a_1^4. \end{aligned}\right\} \tag{16}$$

The solution IV^- does not represent a minimum of $\delta\mathscr{U}_3$.

Since, however, in the present problem, $2\beta(120^\circ) - \beta(0) = 0{\cdot}049 > 0$, the solution IV^+ does provide an energy minimum if

$$-(4T)^{-1} < \epsilon/\gamma^2 < 2(\beta(120^\circ) + \beta(0))\,(2\beta(120^\circ) - \beta(0))^{-2}, \tag{17}$$

or

$$-0{\cdot}241 < \epsilon/\gamma^2 < 410{\cdot}26.$$

[For the opposite case $(2\beta(120^\circ) - \beta(0) < 0)$ the minimum condition (17) is different: $-(4T)^{-1} < \epsilon/\gamma^2 < \infty$.]

Equations (15) also have some other solutions. All of them may, however, be obtained from I–IV by exchanging the positions of the amplitudes a_1, a_2 and a_3 and altering the signs of any two amplitudes simultaneously.

Solutions I–IV are listed together with the stationary energies and conditions for minima. The two limits in (17) are calculated from the equation

$$\det\,(\partial^2 \delta\mathscr{U}_3/\partial a_i\, \partial a_j) = 0.$$

Instability for solutions III and IV^- follows from the listed expressions for $\delta\mathscr{U}_3$. On substituting $\gamma \cos\delta$ in these instead of γ for $\epsilon > 0$, it is easy to see that $\delta\mathscr{U}_3$ as a function of δ has a maximum. For $\epsilon < 0$, solution III is meaningless, but for IV^-, $\delta\mathscr{U}_3$ has a maximum as a function of a_1.

Note that solutions of the form I–IV were obtained previously by Segel & Stuart (1962) for hexagonal cells in thermal convection. They started from the equations of motion and obtained equilibrium equations equivalent to (15) with the same solutions. Their stability condition for IV^+ is in agreement with (17) but their conditions for solutions II, III and IV^- differ from ours. Their stability criterion was found by assuming $a_2 = a_3$ and $\delta = 0$ and taking into account only coupling between three waves at an angle of 120° to one another. In our problem, solution II is unstable with respect to the development of a perpendicular wave, but III and IV^- are unstable with respect to the variation of the phase δ.

For solution IV^+, the amplitudes are always greater than $\gamma/(2T)$. This solution is valid for our problem only if the amplitudes are small. If this is not satisfied then the terms omitted from (4) must be taken into account. This provides a necessary condition for the validity of the above analysis: $\gamma \ll 1$, i.e. $|\mu - 1| \ll 1$.

4.4. *Stability of square and hexagonal arrays*

In §§4.2 and 4.3 it was noted that, with respect to variations of particular waves, the square array is stable for $\epsilon > 0$ and the hexagonal array under the condition (17). To investigate whether these arrays are stable with respect to other perturbations, we assume that there are n (= 2 or 3) dominant main waves with equal amplitudes A and that the amplitudes of the other $N - n$ main waves (perturbations) are much smaller ($|a_{\kappa_j}| \ll A$ if $n < j \leqslant N$). Defining $\delta\mathscr{U}_n$ as the energy of the dominant waves alone, from (5) we may obtain

$$\delta\mathscr{U}_N - \delta\mathscr{U}_n \approx \sum_{j=n+1}^{N} \left(-\tfrac{1}{2}\epsilon + A^2 \sum_{i=1}^{n} \beta(\theta_{ij}) \right) a_{\kappa_j}^2 - \gamma A \sum_{\substack{i \leqslant n < j < l \leqslant N \\ \kappa_i \pm \kappa_j \pm \kappa_l = 0}} a_{\kappa_j} a_{\kappa_l} \cos(\delta_i \pm \delta_j \pm \delta_l).$$

From this expression it follows that, under the condition (17), the hexagonal array is stable because no small perturbation can make the difference $\delta\mathscr{U}_N - \delta\mathscr{U}_n$ negative.

For the square array the most important perturbation consists of two waves with equal amplitudes forming angles of 120° with one of the dominant waves. For

$$\epsilon\gamma^{-2} < (\beta(0) + 2\beta(90°))\,(2\beta(60°) + 2\beta(30°) - 2\beta(90°) - \beta(0))^{-2} = 7{\cdot}438$$

such a perturbation decreases the energy and transforms the square array into the hexagonal array. Otherwise the square array is stable and its amplitude

$$A = |a_{\kappa_1}| = |a_{\kappa_2}|$$

is always larger than

$$A_{\min} = (2\beta(60°) + 2\beta(30°) - 2\beta(90°) - \beta(0))^{-1}\gamma = 3{\cdot}516\gamma.$$

This is ~ 7 times larger than the corresponding value $\gamma/(2T)$ for the hexagonal array. Therefore the condition $|\mu - 1| \ll 1$ is even more necessary for the square array than for the hexagonal one.

5. Hysteresis

The above analysis shows that the surface has three possible configurations of stable equilibrium: a flat surface, an array of hexagonal waves (16) and an array of square waves (14). The situation is illustrated by the sketch in figure 2 (the drawing

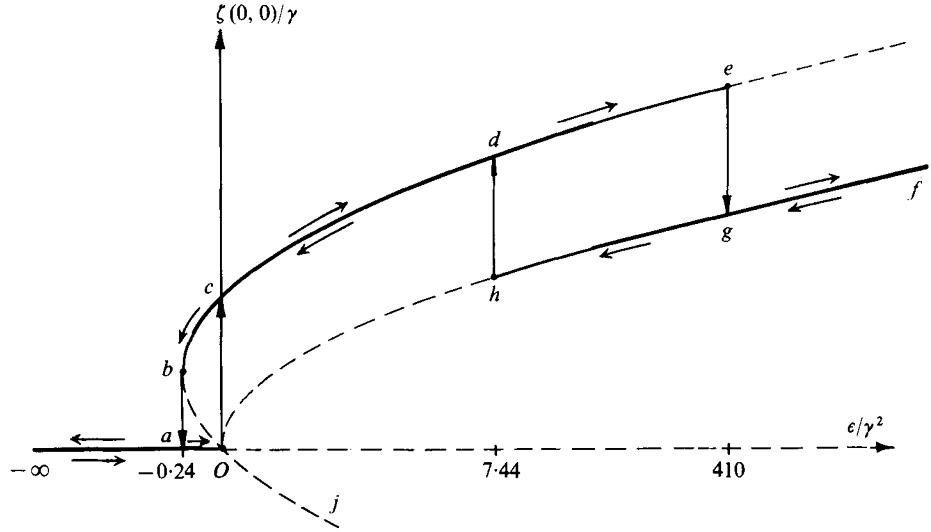

FIGURE 2. The three possible surface configurations corresponding to stable equilibrium.

is made on a deformed scale). There, the largest deviation $\zeta(0,0)$ from a flat surface is shown as a function of $\epsilon = B_0^2/B_c^2 - 1$ (more precisely $\zeta(0,0)/\gamma$ is shown as a function of $\epsilon\gamma^{-2} \approx (4\mu_0)^{-1}(\alpha\rho g)^{-\frac{1}{2}}(B_0^2 - B_c^2)$). The flat surface is represented by the ϵ axis, the hexagonal array by the parabola *jObcde*, and the square array by the parabola *Ohgf*. The absolute minimum of energy is shown by a heavy line, a relative minimum by a light solid line, and an unstable equilibrium by a dashed line.

The square array is represented by only one branch of the parabola *Ohgf* because the other (the symmetric one in the lower half-plane) represents the same surface, but relative to a different origin.

The hexagonal array may be of many different types (see Christopherson 1940). The upper half-plane (curve *Obcde*) represents the array with one crest, two troughs and three saddle points in each elementary hexagonal cell. The lower half-plane (curve *Oj*) represents another type, with two crests, one trough and three saddle points in each elementary cell. In stable equilibrium there is only the first type (line *bcde*). This result is in agreement with the observations made by Cowley & Rosensweig (1967).

It is significant that for some values of ϵ ($\epsilon\gamma^{-2} < -0{\cdot}24$, $0 < \epsilon\gamma^{-2} < 7{\cdot}438$ and $410{\cdot}2 < \epsilon\gamma^{-2}$) the surface has one configuration of stable equilibrium, but at others ($-0{\cdot}24 < \epsilon\gamma^{-2} < 0$ and $7{\cdot}438 < \epsilon\gamma^{-2} < 410{\cdot}2$) it has two such configurations. Which of these two configurations is actually realized depends on how the equilibrium is established. Hence hysteresis phenomena are to be expected.†

Figure 2 allows us to follow the form of surface in a magnetic field which changes adiabatically with time, i.e. so slowly that the surface at any moment is in stable equilibrium, and undergoes a transition from one equilibrium state to another when the former becomes unstable. As the field grows, the surface follows the line $-\infty aOcdegf$ from the flat surface to the hexagonal array and then to the square array. As the field

† This type of hysteresis is of course totally distinct from any ferromagnetic hysteresis inside the ferromagnetic particles forming the fluid.

decreases the surface changes in the opposite sequence, but by a different route, viz. $fghdcba - \infty$. The transition Oc from a flat surface to a hexagonal array occurs at the critical field ($\epsilon = 0$) but, once formed, the hexagonal array remains stable in some subcritical region and undergoes a transition ba back to the flat state only at $\epsilon = -0{\cdot}24\gamma^2$. In like manner, the transition eg from a hexagonal to a square array takes place at larger ϵ than the opposite transition hd.

In a field varying periodically in time, the system may describe two hysteresis loops $aOcb$ and $degh$. Although in the figure they are drawn as having comparable magnitudes, they in fact differ by a factor of ~ 1660 (in the ϵ direction). The possibility of observing both in one test seems unlikely because experimentally conflicting conditions would be necessary.

For the transition eg to be as indicated, it must occur at small wave amplitudes. Therefore the difference $\mu - 1$ must be small enough. To separate the transition Oc from ba, the difference $\mu - 1$ need not be so small, but ϵ must be constant to a high degree over the whole surface. This calls for a very uniform magnetic field, and the smaller $\mu - 1$ is, the greater the need for uniformity.

Appendix. Derivation of the formula (4)

To obtain the expansion in powers of a_{κ_i} for the energy (2), first (i) the field $B(x,y,z)$ must be expanded in a series such that the boundary conditions on the surface

$$z = \zeta(x,y) = \sum_{i=1}^{M} a_{\kappa_i} \cos(\boldsymbol{\kappa}_i . \mathbf{r} + \delta_i)$$

are satisfied. Next (ii) the expansions for $\mathbf{B}(x,y,z)$ and $\zeta(x,y)$ must be substituted into (2) and a non-averaged energy series produced. Finally, (iii) this series must be averaged over x and y. Before doing such a complicated calculation, it is possible to establish from symmetry alone what the averaged expansion (4) must look like.

Although in § 3 the more compact notation of (7) is used, the symmetry may more easily be seen if in steps (i) and (ii) the various series are written in the form

$$\ldots + \underbrace{\sum_{i=1}^{M} \ldots \sum_{j=1}^{M}}_{L} \underbrace{C(\boldsymbol{\kappa}_i, \ldots, \boldsymbol{\kappa}_j)}_{L} \times \underbrace{a_{\kappa_i} \begin{Bmatrix} \cos(\boldsymbol{\kappa}_i . \mathbf{r} + \delta_i) \\ \sin(\boldsymbol{\kappa}_i . \mathbf{r} + \delta_i) \end{Bmatrix} \times \ldots \times a_{\kappa_j} \begin{Bmatrix} \cos(\boldsymbol{\kappa}_j . \mathbf{r} + \delta_j) \\ \sin(\boldsymbol{\kappa}_j . \mathbf{r} + \delta_j) \end{Bmatrix}}_{L \text{ times}} + \ldots, \quad \text{(A 1)}$$

where all Lth-order terms are divided into a sufficient number of subterms that, in each, any a_{κ_i} is accompanied by the one of two possible multipliers: $\cos(\boldsymbol{\kappa}_i . \mathbf{r} + \delta_i)$ or $\sin(\boldsymbol{\kappa}_i . \mathbf{r} + \delta_i)$. After averaging (A 1) there remain only terms for which

$$\boldsymbol{\kappa}_i \pm \ldots \pm \boldsymbol{\kappa}_j = 0. \quad \text{(A 2)}$$

It follows that (4) contains no first-order terms, and that second-order terms are represented by squares $a^2_{\kappa_i}$ only. There are two kinds of fourth-order term ($\sim a^2_{\kappa_i} a^2_{\kappa_j}$ and $\sim a^4_{\kappa_i}$) and four kinds of third-order term: $\sim a^2_{\kappa_i} a_{2\kappa_i}$, $\sim a_{\kappa_i} a_{\kappa_j} a_{\kappa_i + \kappa_j}$, $\sim a_{\kappa_i} a_{\kappa_j} a_{\kappa_i - \kappa_j}$ and $\sim a_{\kappa_i} a_{\kappa_j} a_{\kappa_l}$. The last is of greatest importance, and it appears if

$$\boldsymbol{\kappa}_i \pm \boldsymbol{\kappa}_j \pm \boldsymbol{\kappa}_l = 0. \quad \text{(A 3)}$$

The coefficients of the terms in (4) may be obtained only by direct calculation. However, from symmetry, $E(B_0, |\boldsymbol{\kappa}|)$ at a^2_{κ} must depend on $|\boldsymbol{\kappa}|$ but not on the wave-vector orientation. $K(\theta_{ij})$ and $Q(\theta_{ij})$ depend only on the angle θ_{ij} between the two unit vectors κ_i and κ_j. Symmetry arguments also determine the coefficients of $a^4_{\kappa_i}$ and $a^2_{\kappa_i}a_{2\kappa_i}$ as $\frac{1}{4}K(0)$ and $\frac{1}{2}Q(0)$ and give the phase multiplier

$$\cos(\delta_i \pm \delta_j \pm \delta_l) \tag{A 4}$$

in the second sum in (4). [The two independent signs $\pm$ in (A 4) are the same as the corresponding signs in (A 3)]. These three results follow from the following sine parity rule:

> For a scalar or the z component of a vector any term in the expansion (A 1) contains only an even number of sine factors (if any). The expansions for the x and y components of vectors (except wave vectors $\mathbf{\kappa}_i$) contain an odd number of sine factors.

The source of this parity rule is the expression (3) for the scalar ζ, which contains only cosines. The parity rule is conserved in all operations used in steps (i) and (ii). There is no direct multiplication by $\mathbf{\kappa}_i$: all vectors are produced by differentiation, which changes one cos to sin (for instance, $\operatorname{grad}\cos(\mathbf{\kappa}.\mathbf{r}+\delta) = -\kappa\sin(\mathbf{\kappa}.\mathbf{r}+\delta)$), or one sin to cos. Vectors are converted back to scalars by the second differentiation or by multiplication with another vector. In both cases sine factors are produced in even numbers. The parity is also preserved when the expansion is multiplied by a scalar, when exponents or roots are expanded, and when products of sine and cosine factors are expanded as a sum or vice versa.

We must therefore average only products with an even number of sine factors. At third order we have two such products:

$$\left.\begin{aligned} &\overline{\cos(\mathbf{\kappa}_i.\mathbf{r}+\delta_i)\cos(\mathbf{\kappa}_j.\mathbf{r}+\delta_j)\cos(\mathbf{\kappa}_l.\mathbf{r}+\delta_l)} = \tfrac{1}{4}\cos(\delta_i \pm \delta_j \pm \delta_l),\\ &\mathbf{\kappa}_i.\mathbf{\kappa}_j\,\overline{\cos(\mathbf{\kappa}_i.\mathbf{r}+\delta_i)\sin(\mathbf{\kappa}_j.\mathbf{r}+\delta_j)\sin(\mathbf{\kappa}_l.\mathbf{r}+\delta_l)} = \tfrac{1}{8}(\kappa_i^2-\kappa_l^2-\kappa_j^2)\cos(\delta_i \pm \delta_j \pm \delta_l)\\ &\text{(if (A 3) is satisfied).}\end{aligned}\right\} \tag{A 5}$$

Both give the same multiplier (A 4) as in (4). In (3) the phases of the additional waves are so related that for $a^2_{\kappa_i}a_{2\kappa_i}$ and $a_{\kappa_i}a_{\kappa_j}a_{\kappa_i\pm\kappa_j}$ terms the coefficient (A 4) is equal to 1.

The relation between the coefficients of $a^2_{\kappa_i}a^2_{\kappa_j}$ and $a^4_{\kappa_j}$ may be established if only two waves a_{κ_1} and a_{κ_2} with $\theta_{12} \ll 1$ are treated. In all intermediate series a_{κ_1} and a_{κ_2} must appear symmetrically, and the only acceptable form for the fourth-order energy terms is the following:

$$\begin{aligned}\mathscr{U}^{(4)} = {} & C_1\overline{[a_{\kappa_1}\cos(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)+a_{\kappa_2}\cos(\mathbf{\kappa}_1.\mathbf{r}+\delta_2)]^4}+C_2\overline{[a_{\kappa_1}\sin(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)}\\ & \overline{+a_{\kappa_2}\sin(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)]^4}\\ & +C_3\overline{[a_{\kappa_1}\cos(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)+a_{\kappa_2}\cos(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)]^2[a_{\kappa_1}\sin(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)}\\ & \overline{+a_{\kappa_2}\sin(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)]^2}+a^2_{\kappa_1}a^2_{\kappa_2}O(\theta^2_{12}).\end{aligned} \tag{A 6}$$

Using the results

$$\begin{aligned}&\overline{\cos^2(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)\cos^2(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)} = \overline{\sin^2(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)\sin^2(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)}\\ &\quad = 2\,\overline{\cos(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)\sin(\mathbf{\kappa}_1.\mathbf{r}+\delta_1)\cos(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)\sin(\mathbf{\kappa}_2.\mathbf{r}+\delta_2)} = \tfrac{1}{4},\\ &\quad \overline{\cos^4(\mathbf{\kappa}.\mathbf{r}+\delta)} = \overline{\sin^4(\mathbf{\kappa}.\mathbf{r}+\delta)} = 3\,\overline{\cos^2(\mathbf{\kappa}.\mathbf{r}+\delta)\sin^2(\mathbf{\kappa}.\mathbf{r}+\delta)} = \tfrac{3}{8},\end{aligned}$$

it follows that

$$\mathscr{U}^{(4)} = (\tfrac{3}{2}C_1+\tfrac{3}{2}C_2+2C_3)\,[a^2_{\kappa_1}a^2_{\kappa_2}+\tfrac{1}{4}(a^4_{\kappa_1}+a^4_{\kappa_2})]+a^2_{\kappa_1}a^2_{\kappa_2}O(\theta^2_{12}).$$

For any values of C_1, C_2 and C_3 in the limit $\theta_{12} \to 0$, the coefficient of $a^2_{\kappa_1} a^2_{\kappa_2}$ is four times as large as the coefficient of $a^4_{\kappa_1}$. Similarly, the coefficient of $a^2_{\kappa_i} a_{2\kappa_i}$ may be found to be $-\frac{1}{2}Q(0)$.

These considerations are in full agreement with the results of direct calculation.

REFERENCES

Bénard, H. 1901 *Ann. Chim. Phys.* **23**, 62.

Christopherson, D. G. 1940 *Quart. J. Math.* **11**, 63.

Cowley, M. D. & Rosensweig, R. E. 1967 *J. Fluid Mech.* **30**, 671.

Frenkel, J. I. 1935 *Phys. Zs. Sowjetunion* **8**, 675.

Gailītis, A. 1969 *Magnitnaya Gidrodinamika* **1**, 68.

Melcher, J. R. 1963 *Field Coupled Surface Waves.* M.I.T. Press.

Palm, E. 1960 *J. Fluid Mech.* **8**, 183.

Pellew, A. & Southwell, R. 1940 *Proc. Roy. Soc.* A **176**, 312.

Rayleigh, Lord 1916 *Scientific Papers*, vol. 6, p. 432. Cambridge University Press.

Segel, L. A. & Stuart, J. T. 1962 *J. Fluid Mech.* **13**, 289.

Taylor, G. I. & McEwan, A. D. 1965 *J. Fluid Mech.* **22**, 1.

Zaitsev, V. M. & Shliomis, M. I. 1969 *DAN S.S.S.R.* **188**, 1261.